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Scheme for Efficient Construction of 2-Dimensional Qutrit Cluster State

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Abstract Utilizing quantum entangling gates with an arbitrary small success probability p, we propose a three-step scheme for conclusively constructing 2-dimensional (2D) lattice qutrit cluster states (QTCS). We first prepare one-dimensional (1D) QTCS, then use the 1D states to generate cross-shaped QTCS, and finally take the cross-shaped QTCS as "building blocks" to construct 2D lattice QTCS. In our scheme parallel operations are executed so that the construction process is significantly accelerated. Our scheme provides a potential efficient way to implement realistic one-way quantum computers.

Keywords Cluster state · Qutrit · Lattice · Quantum computer

1 Introduction

Quantum computer have tremendous computational ability. It can undertake formidable computational tasks which are intractable for classical computer. This is one of the reasons why the research of quantum computer has been gone up rapidly in recent years [1]. In most current efforts, universal quantum computation is achieved with sequences of controlled interactions between selected qubits [2–7]. Significantly differently from these efforts, Raussendorf and Briegel [8–10] proposed a new kind of scalable quantum computation, namely, one-way quantum computation, which constructs quantum logic gates by single particle measurements on cluster states. The distinct advantage of the one-way computing strategy lies in that it separates the processes of generating entanglement and executing the computation [11, 12]. So one can tolerate failures during the generation process simply by repeating the process, provided that the failures are heralded. In this model, a two-dimensional (2D) lattice qubit cluster state (QBCS) with numerous qubits must be prepared in advance, which serves as a "substrate" for the computation, then the remaining work is

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to perform single qubit measurements, the final results are read out from those qubits that were not measured in the whole process.

On this background, the efficient construction of 2D or higher dimensional cluster state is significant [13–16]. The main problem of construction of 2D or higher dimensional cluster state is that the successful probability of two-qubit joint operation is so small,which is the case for current experimental systems [17–21]. So one cannot efficiently construct high dimensional cluster state as the overall success probability scales down exponentially as p^n with the number n of the joint operations. In the past few years, some schemes have been proposed to resolve the scaling problem in qubit system. Barrett and Kok [22] presented efficient schemes to construct 2D QBCS from T pieces. In 2005, Duan et al. presented a scheme to efficiently construct 2D lattice-like QBCS with probabilistic quantum gates [13]. In this scheme the scalability of the cluster state preparation is improved. Soon after the scheme of Duan, Chen et al. presented a new scheme [14] based on the idea of Duan's to show that an arbitrary 2D cluster state can be constructed from star-like cluster state and the scalability of the cluster state preparation is even better.

But up to now, there is no articles, as we know, discuss the efficient construction of 2D or higher dimensional cluster state in qutrit systems or in qudit systems, not only from the theoretical point of view, but also on the physical realization of thus system. Naturally, the qutrit or qudit systems have larger Hilbert space which can be used for quantum computation than qubit system [23, 24], the qutrit cluster state (QTCS) or qudit cluster states (QDCS) will have more powerful computational ability than QBCS with the same number of basic unit. Early in 2003, Zhou et al. [15] suggested quantum computation in qudit system and proved the universality of the qudit cluster state computer. However without the efficient construction of 2-dimensional qudit cluster state, qudit cluster state computer should have no meanings. Considering this point, we here propose a scheme for efficient construction of 2-dimensional qutrit cluster state. Naively, we assume that one can reliably perform two-qutrit control operation, $U_{AB}^{(3)}$ operation, with a small success probability p. One can efficiently prepare the 2D qutrit cluster states with the small success probability p by the method of our scheme. Efficient preparation of 2D qutrit cluster states, together with single-qutrit operations, realizes universal quantum computation [11, 12].

2 Construction of 2-Dimensional Lattice Qutrit Cluster State

In this section we will amply explain how to efficiently construct 2D lattice qutrit cluster states (QTCS). In our scheme some properties of QTCS are needed and we first enumerate these useful properties in Sect. 2.1. Then we elaborate how to efficiently construct 1-dimensional (1D) QTCS, cross-shaped QTCS and 2D lattice QTCS in Sect. 2.2, Sect. 2.3 and Sect. 2.4 respectively.

2.1 Useful Properties of Qutrit Cluster State

One can get the properties of QTCS from its definition. The 1D QTCS chains can be described as [15]:

$$|\psi\rangle_{A_1A_2\cdots A_N} = \frac{1}{3^{N/2}} \bigotimes_{j=1}^{N} (|0\rangle + |1\rangle Z_{A_{j+1}} + |2\rangle Z_{A_{j+1}}^2)_{A_j}, \tag{1}$$

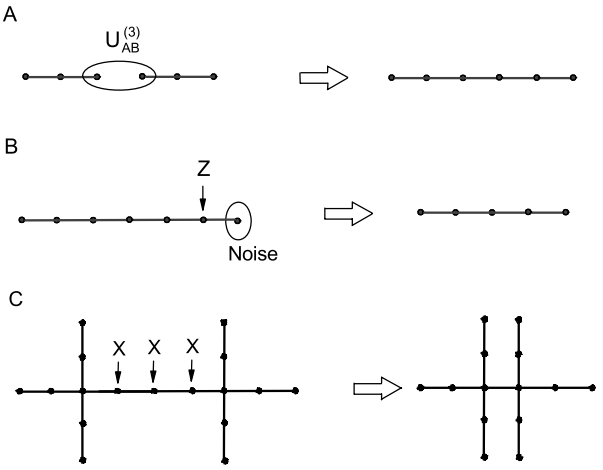

Fig. 1 Illustration of the three properties of the QTCS: (**A**) extend QTCS with $U_{AB}^{(3)}$ operations; (**B**) recover QTCS by removing bad qutrits; (**C**) shrink QTCS for more complicated links

which is the common eigenstate of eigenvalue 1 of the operators $X_i^{\dagger} \bigotimes_{j\in N(i)} Z_j$, i.e.,

$$X_i^{\dagger} \bigotimes_{j\in N(i)} Z_j|\psi\rangle = |\psi\rangle, \quad \text{where } \mathcal{N}(i) = \begin{cases} \{2\}, & i = 1, \\ \{N-1\}, & i = N, \\ \{i-1, i+1\}, & i \notin \{1, N\}. \end{cases}$$

The forms of operators X and Z which are so-called "generalized Pauli operators" [25–27] are given below. They are named in this way because of the convenience of contrasting with the corresponding operators in qubit system.

$$X = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle\langle 0| \tag{2}$$

$$Z = |0\rangle\langle 0| + e^{i\frac{2\pi}{3}}|1\rangle\langle 1| + e^{i\frac{4\pi}{3}}|2\rangle\langle 2| \tag{3}$$

$$U_{AB}^{(3)} = \sum_{m=0}^{2}\sum_{n=0}^{2} e^{i\frac{2\pi}{3}m\cdot n}|m\rangle_A\langle m| \otimes |n\rangle_B\langle n|. \tag{4}$$

In our construction of 2D lattice QTCS by the probabilistic $U_{AB}^{(3)}$ operation, we'll make use of the following three properties of QTCS: (i) If we have two chains of QTCS each with n qutrits, we can join them to form a 1D QTCS of $2n$ qutrits by successfully applying a $U_{AB}^{(3)}$ operation on the end qutrits of the two chains. (ii) If we destroy the state of an end qutrit of an n-qutrit QTCS chain, for instance, through an unsuccessful attempt of the $U_{AB}^{(3)}$ operation, we can remove this bad qutrit by performing a qutrit Z measurement on its neighboring qutrit, and recover a QTCS of $n-2$ qutrits. (iii) We can shrink a QTCS by performing qutrit X measurements on all the connecting qutrits. These three properties of the QTCS can be conveniently explained from their above definition [15].

2.2 Preparation of One-Dimensional Qutrit Cluster State

Assume we will construct a 1D QTCS (the so-called 1D QTCS chain) with length n, we must prepare n particles in the state $|\phi\rangle_{A_j} = (|0\rangle + |1\rangle + |2\rangle)_{A_j}$, then applying the $U_{AB}^{(3)}$

operations on the neighboring qutrits successively, after $(n-1)$ times $U_{AB}^{(3)}$ operations, we get a 1D QTCS chain of length n. It can be described by (5).

$$|\psi\rangle_{A_1A_2\cdots A_n} = \frac{1}{3^{n/2}}\bigotimes_{i=1}^{n-1} U_{A_iA_{i+1}} \bigotimes_{i=1}^{n}(|0\rangle+|1\rangle+|2\rangle)_{A_i}$$
$$= \frac{1}{3^{n/2}}\bigotimes_{j=1}^{n}(|0\rangle+|1\rangle Z_{A_{j+1}}+|2\rangle Z_{A_{j+1}}^2)_{A_j}. \tag{5}$$

These process is conceived in quantum mechanics theory, but considering a small successful probability of $U_{AB}^{(3)}$ operation, the probability of successfully construct a 1D QTCS chain of length n (n is a large number) closes to zero. In order to efficiently construct 1D QTCS chain, based on Duan's idea [13], we divide the construction process into two steps: (1) generate cluster chains longer than some critical length n_c, (2) connect them through a probabilistic $U_{AB}^{(3)}$ operation. These two steps look like very easy, in fact they are onerous because we must repeat too much times before we can get one successful operation. The generation of cluster chains with length n_c is indispensable and n_c is a value get from the statistical calculation for the construction process. In order to make it clearly we assume we have generated two sufficiently long cluster chains each of n_0 qutrits, we try to connect them through a probabilistic $U_{AB}^{(3)}$ operation. If this attempt fails, through the property (ii), we can recover two (n_0-2)-qutrit QTCS chains through a Z measurement, and try to connect them again. As one continues with this process, the average number of the qutrits in the connected chains is then given by $n_l = \sum_{i=0}^{n_0/2} 2(n_0-2i)p(1-p)^i \simeq 2n_0 - 4(1-p)/p$, where the last approximation is valid when $e^{-n_0p/2} \ll 1$. So the average chain length goes up if $n_0 > n_c \equiv 4(1-p)/p$. To show the advantage of this construction process we can calculate the computation overhead scales in terms of the total computation time and the total number of attempts. For the rth ($r>1$) round of successful connection, the chain length n_r, the total preparation time T_r, and the total number of attempts M_r scale up, respectively, by the recursion relations $n_r = 2n_{r-1} - n_c$, $T_r = T_{r-1} + (t_a+\tau_x)/p$ and $M_r = 2M_{r-1} + 1/p$. In the above recursion relations, we have assumed that two QTCS chains for each connection are prepared in parallel, t_a denotes the time for each attempt of the $U_{AB}^{(3)}$ operation, and τ_x denotes the time for single-qutrit X measurement. From these recursion rules, we get $n_r = (n_0-n_c)2^r + n_c$, $T_r = T_0 + r(t_a+\tau_x)/p$, and $M_r = (M_0+1/p)2^r - 1/p$, where T_0 and M_0 stand for the time and attempts needed for a QTCS of length n_0. According to the first formula, r can be expressed with the main chain length n (n_r) after rth rounds successful connections, which can be written as $r = \log_2[(n-n_c)/(n_0-n_c)]$. For a large cluster state with the chain length n we can give the preparation time T and the number of attempts M scale as $T(n) = T_0 + (1/p)(t_a+\tau_x)\log_2[(n-n_c/(n_0-n_c)]$ and $M(n) = (M_0+1/p)(n-n_c)/(n_0-n_c) - 1/p$.

From above analysis we can see that if one can prepare QTCS chains longer than some critical length n_c, we can generate large scale 1D QTCS very efficiently. Then how we can efficiently prepare the QTCS chains up to the critical length n_c. Based on Duan's idea, we propose to use a repeater protocol which divides the task into $m = \log_2 n$ steps: For the ith ($i = 1, 2, \ldots, m.$) step, we attempt to build a $2i$-QTCS by connecting two $2(i-1)$-qutrit QTCS chains through a probabilistic $U_{AB}^{(3)}$ operation. If such an attempt fails, we discard all the qutrits and restart from the beginning. Certainly we may not discard all the qutrits, but instead do some repairing and reconnection as described in the previous paragraph, the protocol will be somewhat more efficient. But for $n < n_r$, such repairing has no significant influence on the basic scaling rules. For the ith step, the recursion relations for the

preparation time T_i and the number of attempts M_i are given by $T_i = (1/p)(T_{i-1} + t_a)$ and $M_i = (1/p)(2M_{i-1} + 1)$, which, together with $T_1 = t_a/p$ and $M_1 = 1/p$, give the scaling rules $T(n) \simeq t_a(1/p)^{\log_2 n}$ and $M(n) \simeq (2/p)^{\log_2 n}/2$. Considering the computation overhead of the repeater protocol, one can find the preparation time T and the number of attempts M scale with the chain length n as $T(n) \simeq t_a(1/p)^{\log_2 n}$ and $M(n) \simeq (2/p)^{\log_2 n}/2$. If one uses a direct protocol to construct such a QTCS chain of length n the preparation time T' and the number of attempts M' will be $T'(n) = t_a(\frac{1}{p})^{n-1}$, $M'(n) = (\frac{1}{p})^{n-1}$. By dividing the task into a series of independent pieces, we improve the scaling with n from exponential to polynomial (for $n < n_c$).

To generate a QTCS chain of a length $n > n_c$, we simply combine the above two protocols. First, we use the repeater protocol to generate n_0-qutrit QTCS chains with $n_0 > n_c$. Then it is straightforward to use the connect-and-repair protocol to further increase its length. For instance, with $n_0 = n_c + 1$, the overall scaling rules for T and M are (for $n > n_c$),

$$T(n) \simeq t_a(1/p)^{\log_2(n_c+1)} + (1/p)(t_a + \tau_x)\log_2(n - n_c), \tag{6}$$

$$M(n) \simeq (2/p)^{\log_2(n_c+1)}(n - n_c)/2. \tag{7}$$

As the critical length is $n_c \simeq 4/p$, T and M in our protocol scale with $1/p$ as $(1/p)^{\log_2(4/p)}$, which is much more efficient than the super-exponential scaling $(1/p)^{4/p}$ in a direct protocol.

2.3 Generation of Cross-Shaped Qutrit Cluster State

We have shown that for any success probability p of the probabilistic entangling gate, 1D QTCS of arbitrary length can be created efficiently. For universal quantum computation, however, such 1D cluster states are not sufficient. They need to be first connected and transformed into 2D QTCS. So suppose we have prepared a certain quantity of 1D QTCS chains with the length of $2n_l + 1$. In order to construct 2D QTCS, we firstly try to build the cross-shaped cluster state by connecting the middle qutrits of two 1D $(2n_l + 1)$-QTCS chains together. This may be achieved by applying a single-qutrit operation $H^{(3)} = \frac{1}{\sqrt{3}} \sum_{m=0}^{2} \sum_{n=0}^{2} e^{i\frac{2\pi}{3}m\cdot n}|m\rangle\langle n|$ on the middle qutrit of one chain and then a $U_{AB}^{(3)}$ operation on the two middle qutrit of the two chains to connect the two middle qutrits and finally an X measurement on one middle qutrit, as is illustrated in Fig. 2. With an average $1/p$ repetitions, we succeed in constructing a cross-shaped cluster state with 4 'legs' of length n_l, which will act as the basic structure unit in the construction of 2-dimensional lattice QTCS.

2.4 Construction of 2-Dimensional Lattice Qutrit Cluster State

We'll use the cross-shaped QTCS as the "building blocks" to construct 2-dimensional lattice QTCS. The leg qutrits serve as ancilla to generate near deterministic connection from the probabilistic $U_{AB}^{(3)}$ operations. The critical idea here is that if we want to connect two center qubits, we always start the connection along the end qutrits of one of the legs. If such an attempt fails, we can delete two end qutrits and try the connection again along the same legs. If the leg is sufficiently long, we can almost certainly succeed before we reach (destroy) the center qutrits. When we succeed, and if there are still redundant leg qutrits between the two center ones, we can delete the intermediate leg qutrits by performing simple single-qutrit X measurements on all of them. With such a procedure, we can continuously connect the center qutrits and form any complex lattice geometry.

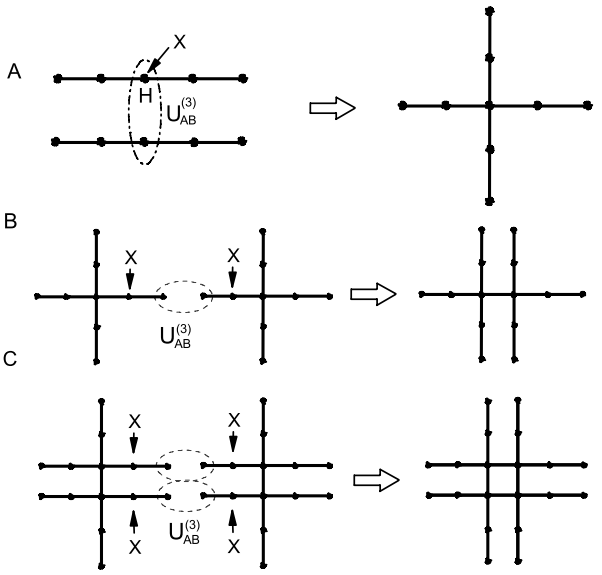

Fig. 2 Illustration of the steps for construction of the two-dimensional square lattice QTCS from a set of cluster chains. (**A**) Construction of the basic cross-shaped states from qutrit cluster chains by applying first a $H^{(3)}$ operation on the middle qutrit of one chain, and then a $U_{AB}^{(3)}$ operation to connect the two middle qutrits, and finally a X measurement on one middle qutrit to remove it. (**B**,**C**) Construction of the square lattice cluster state from the cross-shaped states through probabilistic $U_{AB}^{(3)}$ operations along the legs and X measurements to remove the remaining redundant qutrits

From the process described above, we can find out that the overall success probability depends on the length of every leg. The longer the legs are, the higher the success probability is. If the legs are long enough, we can almost make the connection of two center qutrits succeed with a probability p near to 1.

Now let us consider the scaling rules of the computational overhead with the 2D QTCS. Assume we want to build a 2D cluster state with qutrit number N. The key process is enumerated below.

(a) Generation of the 1D $2n_l + 1$ qutrit chains. $T(2n_l + 1) \simeq t_a(1/p)^{\log_2(p/4+1)} + (1/p) \times (t_a + \tau_x) \log_2(2n_l - p/4 + 1)$.

(b) Generation of the cross-shaped unit. On average, $1/p$ repetitions, we can succeed in getting a cross-shaped QTCS from two qutrit cluster chains. Thus, the operation time should be $t_a = (t_a + \tau_h + \tau_x)/p$, τ_h denotes the time of $H^{(3)}$ operation. It is worth to point out that if the connection of two chains is unsuccessful, the two chains cannot be used again and must be discarded because of the noise introduced to the cluster states in the construction process. So on average $2/p$ 1D qutrit cluster chains is needed for one time successful connection.

(c) Connect the center qutrits of two cross-shaped QTCS to build 2D QTCS. If the ancillary legs have length n_l, for each connection of two center qutrits, we can try at most $n_l/2$ times of the probabilistic $U_{AB}^{(3)}$ operations, and the overall success probability is given by $p_c = 1 - (1 - p)^{n_l/2}$. If we want to build a square lattice cluster state of N qutrits, we need about $2N$ times of connections of the center qutrits because there are about $2N$ edges in an N-vertex square lattice. The probability for all these connections to be successful is given by p_c^{2N}. We require this overall success probability is sufficiently large with $p_c^{2N} \geq 1 - \epsilon$, where ϵ is a small number characterizing the overall failure probability. From that requirement, we figure out that $n_l \simeq (2/p) \ln(2N/\epsilon)$. To construct a square lattice cluster state of N qutrits, we need to consume N cross-shaped states, and each of the latter requires on average $2/p$ cluster chains with a length of $2n_l + 1$ qutrits. So we need in total $2N/p$ $(2n_l + 1)$-qutrit cluster chains, which can be prepared in parallel with $(2N/p)M(2n_l + 1)$ times $U_{AB}^{(3)}$ attempts within a time period $T(2n_l + 1)$

(see (6) and (7) for expressions of the $T(n)$ and $M(n)$). This gives the resources for preparation of all the basic building blocks (the cross-shaped chains). Then we need to connect these blocks to form the square lattice. We assume that the connection of all the building blocks are done in parallel. The whole connection takes on average $2N/p$ times $U_{AB}^{(3)}$ attempts, and consumes a time at most $t_a/p\ln(2N/\epsilon)$. Summarizing these results, the temporal and the operational resources for preparation of an N qutrit square lattice cluster state are approximately given by

$$T(N) \simeq t_a(1/p)^{\log_2(4/p-3)} + \frac{(t_a+\tau_h+\tau_x)}{p}\log_2\left(\frac{4}{p}[\ln(2N/\epsilon)-1]\right) + \frac{t_a+\tau_x}{p}\ln(2N/\epsilon), \tag{8}$$

$$M(N) \simeq (2/p)^{2+\log_2(4/p-3)}N[\ln(2N/\epsilon)-1]+2N/p. \tag{9}$$

In the 2D case, the temporal and the operational overheads still have very efficient scaling with the qutrit number N, logarithmically for $T(N)$ and $N\ln(N)$ for $M(N)$. Their scalings with $1/p$ are almost the same as in the 1D case except an additional factor of $1/p^2$ for $M(N)$. Through some straightforward variations of the above method, it is also possible to efficiently prepare any complicated graph state using probabilistic $U_{AB}^{(3)}$ operations. This shows that in principle we do not need to impose any threshold on the success probability of the $U_{AB}^{(3)}$ operations for efficient quantum computation.

Before ending this paper, we would like to add a few remarks on other sources of noise that have not been taken into account in the above discussions. If each $U_{AB}^{(3)}$ operation has some small additional infidelity error, one might wonder whether such an error scales up with the large number of attempts $M(N)$. That is actually not the case. Most of the $U_{AB}^{(3)}$ attempts have failed, and all the failed $U_{AB}^{(3)}$ operations have no contribution to the final state infidelity. Our scheme is also fit to construct k-dimensional ($k \geq 3$) QDCS.

3 Summary

To summarize, in this paper we have proposed a scheme to construct 2-dimensional QTCS with efficient scaling in both the qutrit number and the inverse of the success probability. An efficient preparation together with simple single-qutrit operations can realize universal qutrit quantum computation. Moreover, this idea can be easily generalized to construct k-dimensional ($k \geq 3$) qudit cluster state in a similar way.

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